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Probabilistic Study of the Speed of Approach to Equilibrium for an Inelastic Kac Model

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Abstract This paper deals with a one-dimensional model for granular materials, which boils down to an inelastic version of the Kac kinetic equation, with inelasticity parameter $p > 0$. In particular, the paper provides bounds for certain distances—such as specific weighted χ-distances and the Kolmogorov distance—between the solution of that equation and the limit. It is assumed that the even part of the initial datum (which determines the asymptotic properties of the solution) belongs to the domain of normal attraction of a symmetric stable distribution with characteristic exponent $\alpha = 2/(1 + p)$. With such initial data, it turns out that the limit exists and is just the aforementioned stable distribution. A necessary condition for the relaxation to equilibrium is also proved. Some bounds are obtained without introducing any extra condition. Sharper bounds, of an exponential type, are exhibited in the presence of additional assumptions concerning either the behaviour, close to the origin, of the initial characteristic function, or the behaviour, at infinity, of the initial probability distribution function.

Keywords Central limit theorem · Domains of normal attraction · Granular materials · Kolmogorov metric · Inelastic Kac equation · Stable distributions · Sums of weighted independent random variables · Speed of approach to equilibrium · Weighted χ-metrics

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1 Introduction

This work deals with a one-dimensional inelastic kinetic model, as introduced in [30], which can be thought of as a generalization of the Boltzmann-like equation due to Kac [27]. Motivations for research into equations for inelastic interactions can be found in many papers, generally devoted to Maxwellian molecules. Among them, in addition to the already mentioned Pulvirenti and Toscani's paper, it is worth quoting: [2–8, 10, 15]. See, in particular, a short but useful review in [32]. Returning to the main subject of this paper, the one-dimensional inelastic model we want to study reduces to the equation

$$\begin{cases} \dfrac{\partial}{\partial t} f(v,t) + f(v,t) = \dfrac{1}{2\pi} \displaystyle\int_{\Re\times[0,2\pi)} f\left(\dfrac{vc(\theta) - ws(\theta)}{J(\theta)}, t\right) f\left(\dfrac{vs(\theta) + wc(\theta)}{J(\theta)}, t\right) \dfrac{dw d\theta}{J(\theta)} \\ f(v,0) := f_0(v) \quad (t > 0, v \in \mathbb{R}) \end{cases} \tag{1}$$

where $f(\cdot,t)$ stands for the probability density function of the velocity of a molecule at time t and

$$c(\theta) := \cos\theta|\cos\theta|^p, \qquad s(\theta) := \sin\theta|\sin\theta|^p, \qquad J(\theta) := c(\theta)^2 + s(\theta)^2,$$

p being a nonnegative parameter. When $p = 0$, (1) becomes the Kac equation. It is easy to check that the Fourier transform $\phi(\cdot,t)$ of $f(\cdot,t)$ satisfies equation

$$\begin{cases} \dfrac{\partial}{\partial t}\phi(\xi,t) = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} \phi(\xi s(\theta),t)\phi(\xi c(\theta),t)d\theta - \phi(\xi,t) \\ \phi(\xi,0) := \phi_0(\xi) \quad (t > 0, \xi \in \mathbb{R}), \end{cases} \tag{2}$$

where ϕ_0 stands for the Fourier transform of f_0.

Equation (2) can be considered independently of (1), thinking of $\phi(\cdot,t)$ as Fourier–Stieltjes transform of a probability measure $\mu(\cdot,t)$, with $\mu(\cdot,0) := \mu_0(\cdot)$. In this case, differently from (1), μ needn't be absolutely continuous, i.e. it needn't have a density function with respect to the Lebesgue measure. In any case, $\mu(\cdot,t)$ is completely determined by its restriction to the class of intervals $\{(-\infty,x], x \in \mathbb{R}\}$, i.e. $F(x,t) := \mu((-\infty,x],t)$. The function $x \mapsto F(x,t)$, defined for every x in $\mathbb{R}$, is usually called (*probability*) *distribution function*.

Due to dissipation, it is known that all finite energy solutions of (2) decay to the Dirac mass at zero, but there are also infinite energy solutions which converge to non-trivial steady states. See, e.g., [30] and the next Theorem 2.1. The present paper aims at investigating the asymptotic behaviour of this kind of solutions.

Following [33], ϕ can be expressed as

$$\phi(\xi,t) = \sum_{n\geq 1} e^{-t}(1-e^{-t})^{n-1}\hat{q}_n(\xi;\phi_0) \quad (t \geq 0, \xi \in \mathbb{R}) \tag{3}$$

where

$$\begin{cases} \hat{q}_1(\xi,\phi_0) := \phi_0(\xi) \\ \hat{q}_n(\xi;\phi_0) := \frac{1}{n-1}\sum_{j=1}^{n-1} \hat{q}_{n-j}(\xi;\phi_0) \circ \hat{q}_j(\xi;\phi_0) \quad (n = 2,3,\ldots) \end{cases} \tag{4}$$

and

$$g_1 \circ g_2(\xi) = \frac{1}{2\pi}\int_0^{2\pi} g_1(\xi c(\theta))g_2(\xi s(\theta))d\theta \quad (\xi \in \mathbb{R})$$

is the so-called *Wild product*. The Wild representation (3) can be used to prove that the Kac equations (1) and (2) have a unique solution in the class of all absolutely continuous probability measures and, respectively, in the class of the Fourier–Stieltjes transforms of *all* probability measures on the Borel sets of $\mathbb{R}$. Moreover, this very same representation, as pointed out by [28], can be reformulated in such a way as to show that $\phi(\cdot, t)$ is the characteristic function of a completely specified sum of real-valued random variables. This represents an important point for the methodological side of the present work. Indeed, thanks to the McKean interpretation, our study will take advantage of methods and results pertaining to the *central limit theorem* of probability theory.

As to the set-up of the paper, in the second part of the present section we provide the reader with preliminary information—mainly of a probabilistic nature—which is necessary to understand the rest of the paper. In Sect. 2 we present the new results, together with a few hints at the strategies used to prove them. The most significant steps of the proofs are contained in Sect. 3, devoted to asymptotics for weighted sums of independent random variables. The methods used in this section are essentially inspired by previous work of Harald Cramér and by its developments due to Peter Hall. See [12, 13, 24]. Completion of the proofs is deferred to the Appendix.

1.1 Probabilistic Interpretation of Solutions of (1)–(2)

It is worth lingering over the McKean reformulation of (4), following [19]. Consider the product spaces

$$\Omega_t := \mathbb{N} \times \mathbb{G} \times [0, 2\pi)^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$$

with $\mathbb{G} = \bigcup_n G(n)$, $G(n)$ being a set of certain *binary trees* with n leaves. These trees are defined so that each node has either zero or two "children", a "left child" and a "right child". See Fig. 1.

Now, equip Ω_t with the σ-algebra

$$\mathcal{F}_t := \mathcal{P}(\mathbb{N} \times \mathbb{G}) \otimes \mathcal{B}([0, 2\pi)^{\mathbb{N}}) \otimes \mathcal{B}(\mathbb{R}^{\mathbb{N}})$$

where, given any set S, $\mathcal{P}(S)$ denotes the power set of S and, if S is a topological space, $\mathcal{B}(S)$ indicates the Borel σ-algebra on S. Define $(\nu_t, \gamma_t, \theta_t, X_t)$, with $\theta_t := (\theta_{t,n})_{n\geq 1}$ and $X_t := (X_{t,n})_{n\geq 1}$, to be the coordinate random variables of Ω_t. At this stage, for each tree in

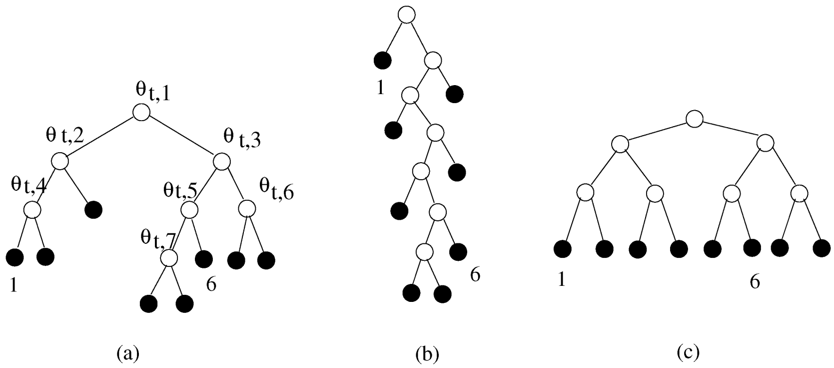

Fig. 1 Example of binary trees. *Shaded* (*unshaded*) *circles* stand for leaves (nodes)

$G(n)$ fix an order on the set of all the $(n-1)$ nodes and, accordingly, associate the random variable $\theta_{t,k}$ with the k-th node. See (a) in Fig. 1. Moreover, call $1, 2, \ldots, n$ the n leaves following a left to right order. See (b) in Fig. 1. Define the depth of leaf j—in symbols, δ_j—to be the number of generations which separate j from the "root" node, and for each leaf j of a tree, form the product

$$\beta_{j,t} := \prod_{i=1}^{\delta_j} \alpha_i^{(j)}$$

where: $\alpha_{\delta_j}^{(j)}$ equals $c(\theta_{t,k})$ if j is a "left child" or $s(\theta_{t,k})$ if j is a "right child", and $\theta_{t,k}$ is the element of θ_t associated to the parent node of j; $\alpha_{\delta_j-1}^{(j)}$ equals $c(\theta_{t,m})$ or $s(\theta_{t,m})$ depending on the parent of j is, in its turn, a "left child" or a "right child", $\theta_{t,m}$ being the element of θ_t associated with the grandparent of j; and so on. For the unique tree in $G(1)$ it is assumed that $\beta_{1,t} = 1$. For instance, as to leaf 1 in Fig. 1(a), $\beta_{1,t} = c(\theta_{t,4}) \cdot c(\theta_{t,2}) \cdot c(\theta_{t,1})$ and, for leaf 6, $\beta_{6,t} = s(\theta_{t,5}) \cdot c(\theta_{t,3}) \cdot s(\theta_{t,1})$.

From the definition of the random variables $\beta_{j,t}$ it is plain to deduce that

$$\sum_{j=1}^{\nu_t} |\beta_{j,t}|^\alpha = 1,$$

holds true for any tree in $G(\nu_t)$, with

$$\alpha := \frac{2}{1+p}.$$

For further information on this construction, see [9, 19, 29].

It is easy to verify that there is one and only one probability measure P_t on $(\Omega_t, \mathcal{F}_t)$ such that

$$\begin{aligned} &P_t\{\nu_t = n, \gamma_t = g, \theta_t \in A, X_t \in B\} \\ &\quad = \begin{cases} e^{-t}(1-e^{-t})^{n-1} p_n(g) u^{\otimes \mathbb{N}}(A) \mu_0^{\otimes \mathbb{N}}(B) & \text{if } g \in G(n) \\ 0 & \text{if } g \notin G(n) \end{cases} \end{aligned}$$

where, for each t,

- p_n is a well-specified probability on $G(n)$, for every n.
- $u^{\otimes \mathbb{N}}$ is the probability distribution that makes the $\theta_{t,n}$ independent and identically distributed with continuous uniform law on $[0, 2\pi)$.
- $\mu_0^{\otimes \mathbb{N}}$ is the probability distribution according to which the random variables $X_{t,n}$ turn out to be independent and identically distributed with common law μ_0.

Expectation with respect to P_t will be denoted by E_t and integrals over a measurable set $A \subset \Omega$ will be often indicated by $E_t(\cdot; A)$.

In this framework one gets the following proposition, a proof of which can be obtained from obvious modifications of the proofs of Theorem 3 and Lemma 1 in [19].

(F_1) *The solution* $f(\cdot, t)$ ($\phi(\cdot, t)$, *respectively*) *of* (1) ((2), *respectively*) *can be viewed as a probability density function* (*the characteristic function, respectively*) *of*

$$V_t := \sum_{j=1}^{\nu_t} \beta_{j,t} X_{t,j}$$

for any $t > 0$. *Moreover,* $\beta_{(\nu_t)} := \max\{|\beta_{1,t}|, \dots, |\beta_{\nu_t,t}|\}$ *converges in distribution to zero as* $t \to +\infty$.

As a first application of this proposition, one easily gets

$$\begin{aligned}\phi(\xi, t) &= E_t[E_t(e^{i\xi V_t}|\nu_t)] \\ &= e^{-t}\phi_0(\xi) + e^{-t}\sum_{n\geq 2}(1-e^{-t})^{n-1}\hat{q}_n(\xi;\phi_0).\end{aligned}$$

Then, since $\hat{q}_n(\xi;\phi_0) = \hat{q}_n(\xi; Re(\phi_0))$ for any $n \geq 2$—with $Re(z) =$ real part of z—the conditional characteristic function of V_t, given $\{\nu_t = n\}$, coincides with the characteristic function of V_t when ϕ_0 is replaced by its real part. Whence,

$$\phi(\xi, t) = e^{-t}\sum_{n\geq 1}(1-e^{-t})^{n-1}\hat{q}_n(\xi; Re(\phi_0)) + i Im(\phi_0(\xi))e^{-t} \tag{5}$$

with $Im(z) :=$ imaginary part of z. The distribution corresponding to $Re(\phi_0)$ is symmetric, or *even part* of μ_0, as it is sometimes labelled. In fact, $Re(\phi_0)$ turns out to be an even real-valued characteristic function, and, generally speaking, this fact does actually simplify certain computations. It should be pointed out that if the initial datum μ_0 is a symmetric probability distribution, then the distribution of V_t is the same as the distribution of $\sum_{j=1}^{\nu_t}|\beta_{j,t}|X_{t,j}$.

1.2 Topics on Stable Distributions

It can be proved that the possible limits (in distribution) of V_t, as $t \to +\infty$, have characteristic functions ϕ which are solutions of

$$\frac{1}{2\pi}\int_0^{2\pi}\phi(\xi s(\theta))\phi(\xi c(\theta))d\theta = \phi(\xi) \quad (\xi \in \mathbb{R}). \tag{6}$$

This result has been communicated to us by Filippo Riccardi, who proved it by resorting to a suitable modification of the Skorokhod representation used in the Appendix of the present paper. We didn't succeed in finding all the solutions of (6), but it is easy to check that

$$\hat{g}_\alpha(\xi) = \exp\{-a_0|\xi|^\alpha\} \quad (\xi \in \mathbb{R}) \tag{7}$$

is a solution of (6), for any $a_0 \geq 0$. In point of fact, for suitable (infinite energy) initial distributions the pointwise limit of the solution of (2) exists and is given by (7), which, in turn, is strictly connected with certain sums of random variables. Indeed, (7) is a *stable real-valued characteristic function* with characteristic exponent α and, in view of a classical Lévy's theorem,

(F_2) *If* $X_1, X_2, \dots$ *are independent and identically distributed real-valued random variables, with symmetric common distribution function* F_0*, then in order that the random variable* X *be the limit in distribution of the normed sum* $\sum_{i=1}^n X_i/n^{1/\alpha}$ *it is necessary and sufficient that* X *has characteristic function* (7) *for some* $a_0 \geq 0$.

It is worth recalling that the probability distribution function, at x, of a random variable Y is defined to be the probability that Y belongs to the interval $(-\infty, x]$, for every real x.

Due to (F_1) one could guess that (F_2) may be used to get a direct proof of the fact that V_t converges in distribution to a stable random variable with characteristic function (7). This way, one would obtain that these characteristic functions are all possible pointwise limits, as $t \to +\infty$, of solutions $\phi(\cdot, t)$ of (2). In point of fact, direct application of results like (F_2) is inadmissible, since V_t is a weighted sum of a random number of summands affected by random weights which are not stochastically independent. In spite of this, by resorting to suitable forms of conditioning for V_t, one can take advantage of classical propositions pertaining to the central limit theorem.

In addition to the problem of determining the class of all possible limit distributions for V_t, an obvious question which arises is that of singling out necessary and sufficient conditions on μ_0, so that it may converge in distribution to some specific random variable. As to the classical setting mentioned in (F_2), it is worth recalling that

(F_3) *If $X_1, X_2, \dots$ are independent and identically distributed real-valued random variables, with (not necessarily symmetric) common distribution function F_0, then in order that $(\sum_{i=1}^n X_i/n^{1/\alpha} - m_n)_{n\geq 1}$ converges in law to a random variable with characteristic function (7) with some specific value for a_0—or, in other words, that F_0 belong to the domain of normal attraction of (7)—it is necessary and sufficient that F_0 satisfies $|x|^\alpha F_0(x) \to c_1$ as $x \to -\infty$ and $x^\alpha(1 - F_0(x)) \to c_1$ as $x \to +\infty$, i.e.*

$$\begin{aligned} &F_0(-x) = \frac{c_1}{|x|^\alpha} + S_1(-x) \quad \textit{and} \quad 1 - F_0(x) = \frac{c_1}{x^\alpha} + S_2(x) \quad (x > 0) \\ &S_i(x) = o(|x|^{-\alpha}) \quad \textit{as} \quad |x| \to +\infty \quad (i = 1, 2). \end{aligned} \tag{8}$$

For more information on stable laws and central limit theorem see, for example, Chap. 2 of [26] and Chap. 6 of [22]. To complete the description of certain facts that will be mentioned throughout the paper, it is worth enunciating

(F_4) *If ϕ_0 stands for the Fourier–Stieltjes transform of a probability distribution function F_0 satisfying (8), then*

$$1 - \phi_0(\xi) = (a_0 + v_0(\xi))|\xi|^\alpha \quad (\xi \in \mathbb{R})$$

where v_0 is bounded and $|v_0(\xi)| = o(1)$ as $|\xi| \to 0$.

(F_4), which is a paraphrase of Théorème 1.3 of [25], can be proved by mimicking the argument used for Theorem 2.6.5 of [26].

2 Presentation of the New Results

In the present paper our aims are: firstly, to find initial distribution functions F_0 (or initial characteristic functions ϕ_0) so that the respective solutions of (2) may converge pointwise to (7); secondly, to determine the rate of convergence of the probability distribution function $F(\cdot, t)$, corresponding to $\phi(\cdot, t)$, to a stable distribution function G_α with characteristic exponent $\alpha = 2/(1+p)$, with respect both to specific weighted χ-metrics and to Kolmogorov's distance, whose definitions are given just below.

It is well-known—from the Lévy continuity theorem—that pointwise convergence of sequences of characteristic functions is equivalent to *weak convergence* of the corresponding distribution functions. In particular, in our present case, since the limiting distribution function G_α is (absolutely) continuous, weak convergence is equivalent to uniform convergence, i.e.

$$\sup_{x\in\mathbb{R}} |F(x, t) - G_\alpha(x)| \to 0 \quad \text{as } t \to +\infty. \tag{9}$$

Left-hand side of (9) is just the *Kolmogorov distance* (K, in symbols) between $F(\cdot, t)$ and G_α. As to the above-mentioned first aim, besides sufficient conditions for convergence—which boil down to the fact that F_0 belongs to the domain of normal attraction of (7)—a necessary condition for convergence is given. As far as rates of convergence are concerned, results can be found in the paper of Pulvirenti and Toscani, with respect to a specific *weighted χ-metric*, used to study convergence to equilibrium of Boltzmann-like equations starting from [21]. See also [31]. Denoting this distance by χ_s, s being some positive number, one has

$$\chi_s(F(\cdot,t),G_\alpha) := \sup_{\xi\in\mathbb{R}} \frac{|\phi(\xi,t)-\exp(-a_o|\xi|^\alpha)|}{|\xi|^s}.$$

With reference to (1), after writing g_α for a density of G_α, Theorem 6.2 in [30] reads:

(F_5) *Let $p>1$ with f_0 such that $\int_{\mathbb{R}}|v|^{\alpha+\delta}|f_0(v)-g_\alpha(v)|dv$ is finite for some δ in $(0,(1-\alpha)\wedge\alpha)$. Then*

$$\chi_{\alpha+\delta}(F(\cdot,t),G_\alpha) \le \chi_{\alpha+\delta}(F_0,G_\alpha)\exp\{-t(1-2A_{2(1+\delta/\alpha)})\} \tag{10}$$

holds true for every $t\ge 0$, with

$$A_m := \frac{1}{2\pi}\int_0^{2\pi}|\sin\theta|^m d\theta = \frac{\Gamma(\frac{m}{2}+\frac{1}{2})}{\sqrt{\pi}\,\Gamma(\frac{m}{2}+1)} \quad (m\ge 0). \tag{11}$$

Moreover, (10) *is still valid if $0<p\le 1$ and $\int_{\mathbb{R}}|v|^{\alpha+\delta}|f_0(v)-g_\alpha(v)|dv$ if finite for some δ in $(0,\alpha p]$.*

It should be pointed out that the proof of (F_5) provided in [30] rests on a hypothesis that is weaker than the one evoked in (F_5), i.e.

$$|v_0(\xi)| = O(|\xi|^\delta) \qquad \text{as } \xi\to 0 \tag{12}$$

for some $\delta>0$.

In the present paper we prove weak convergence of $F(\cdot,t)$ to G_α under much more general hypotheses than those adopted in (F_5). For reader's convenience, it is worth noticing that the probability distribution function F_0^* corresponding to $Re(\phi_0)$ (see the final part of Sect. 1.2) coincides with

$$\frac{1}{2}\{F_0(x)+1-F_0(-x)\}$$

at each point x of continuity for F_0. In view of (F_3)–(F_4), if F_0 belongs to the domain of normal attraction of (7), then there is a nonnegative c_0 for which

$$\lim_{x\to-\infty}|x|^\alpha F_0^*(x) = \lim_{x\to+\infty}x^\alpha(1-F_0^*(x)) = c_0 \tag{13}$$

and the characteristic function associated to F_0^*, i.e. $Re(\phi_0)$, satisfies

$$1-Re(\phi_0(\xi)) = (a_0+v_0^*(\xi))|\xi|^\alpha \tag{14}$$

or some bounded, real-valued v_0^* such that $|v_0^*(\xi)|=o(1)$ as $\xi\to 0$. Moreover, c_0 is related to a_0 by

$$a_0 = 2c_0\int_0^{+\infty}\frac{\sin(x)}{x^\alpha}dx.$$

The precise statement of the aforementioned convergence reads

Theorem 2.1 *Given $p > 0$, let the initial data for problems (1)–(2) satisfy*

$$\lim_{x \to +\infty} (1 - F_0^*(x)) x^\alpha = c_0.$$

Then

$$\lim_{t \to +\infty} K(F(\cdot, t), G_\alpha) = 0.$$

In particular, if $c_0 = 0$, then for every $\epsilon > 0$ one has

$$\lim_{t \to +\infty} F(-\epsilon, t) = \lim_{t \to +\infty} (1 - F(\epsilon, t)) = 0,$$

i.e. the weak limit of $\mu(\cdot, t)$ is the point mass δ_0. On the other hand, if $p > 0$ and there is a strictly positive and increasing sequence $(t_n)_{n \geq 1}$, divergent to $+\infty$, such that $(F(\cdot, t_n))_{n \geq 1}$ converges weakly to any probability distribution function, then

$$0 \leq \lim_{\xi \to +\infty} \inf_{x \geq \xi} x^\alpha (1 - F_0^*(x)) < +\infty.$$

Proof of Theorem 2.1 is deferred to the Appendix.

After presenting the most general statement we achieved about the weak convergence of $F(\cdot, t)$, let us proceed to investigate how convergence is fast. Pulvirenti and Toscani's argument to prove (F_5) lies in studying (4) directly via suitable inequalities and from an analytical viewpoint. In contrast, in our approach one starts from inequality

$$|\phi(\xi, t) - \hat{g}_\alpha(\xi)| \leq E_t(|\tilde{\phi}_{\nu_t}(\xi) - \hat{g}_\alpha(\xi)|) \tag{15}$$

where $\hat{g}_\alpha$ is defined by (7) and, according to (F_1), $\tilde{\phi}_{\nu_t}$ represents the conditional characteristic function of V_t given $(\nu_t, \gamma_t, \theta_t)$. Hence, from the beginning, we try to obtain bounds for $|\tilde{\phi}_{\nu_t}(\xi) - \hat{g}_\alpha(\xi)|$. This is tantamount to investigating bounds for $|\tilde{\phi}_n(\xi) - \hat{g}_\alpha(\xi)|$ when $\tilde{\phi}_n$ is the characteristic function of

$$S_n := \sum_{l=1}^{n} q_l^{(n)} X_l \tag{16}$$

with $X_1, X_2, \ldots$ independent and identically distributed random numbers, with common distribution function F_0, and

$$q_l^{(n)} \geq 0 \quad \text{for } l = 1, \ldots, n, n = 1, 2, \ldots \quad \text{such that } \sum_{l=1}^{n} (q_l^{(n)})^\alpha = 1. \tag{17}$$

Think of n and $(q_1^{(n)}, \ldots, q_n^{(n)})$ as realizations of ν_t and $(|\beta_{1,t}|, \ldots, |\beta_{\nu_t,t}|)$, respectively. According to (F_1) one can assume

$$q_{(n)} := \max\{q_1^{(n)}, \ldots, q_n^{(n)}\} \to 0 \quad \text{as } n \to +\infty. \tag{18}$$

We study this problem—prior to the investigation of rates of convergence for V_t—under the additional conditions that F_0 is *symmetric* (and, consequently, the corresponding characteristic function ϕ_0 is even) and that it belongs to the domain of normal attraction of $\hat{g}_\alpha$. See (F_3)–(F_4) and (13)–(14). This way we also get bounds for convergence in law of weighted sums S_n to stable random variables, which are of interest in themselves and, as far as we know, seem to be new. They are explained and precisely formulated in Sect. 3. Resuming now the main issue of the speed of convergence of V_t to equilibrium, some further notation is needed. We set

$$\|v_0^*\| := \sup_{\xi\geq 0}|v_0^*(\xi)|, \qquad M := a_0 + \|v_0^*\|, \qquad \bar{v}_0^*(\xi) := \sup_{0\leq x\leq \xi}|v_0^*(x)|$$

and, given $\eta \in (0, a_0)$, define d to be some element of $(0, 1)$ such that

$$\frac{4}{5}M^2|x|^\alpha + \bar{v}_0^*(x) \leq \eta$$

comes true whenever $|x| \leq (3d/(8M))^{1/\alpha}$. Next, we put $M_r := \max_{x\geq 0} x^{r\alpha}e^{-(a_0-\eta)x^\alpha}$, $d_1 := (3/(8M))^{1/\alpha}$, $k^* = \bar{v}_0^*(d_1 d^{1/\alpha})(1 + 2d_1^\alpha d^{1-\alpha}\bar{v}_0^*(d_1 d^{1/\alpha})) + (4/5)M^2 d_1^\alpha d + (32/25)M^4 d_1^{3\alpha} d^{3-\alpha}$.

2.1 Speed of Approach to Equilibrium with Respect to Weighted χ-Metrics

Now we are in a position to present our first results which concern convergence of $F(\cdot, t)$ to G_α with respect to χ-metrics.

Theorem 2.2 *Let F_0 belong to the domain of normal attraction of G_α with $\alpha = 2/(1+p)$, for some $p > 0$. Define v_0 and v_0^* to be the same as in (F_4) and (14), respectively. Set $\beta_{(\nu_t)} := \max\{|\beta_{1,t}|, \ldots, |\beta_{\nu_t,t}|\}$. Then*

$$\begin{aligned}\chi_\alpha(F(\cdot,t), G_\alpha) \leq{}& E_t(\bar{v}_0^*(d_1\beta_{(\nu_t)}^c)) + 2M_1 E_t(\bar{v}_0^*(d_1\beta_{(\nu_t)}^c)^2) + \frac{4}{5}M^2 M_1 E_t(\beta_{(\nu_t)}^\alpha)\\ &+ \frac{32}{25}M_3 M^4 E_t(\beta_{(\nu_t)}^{2\alpha}) + \left(k^* + \frac{2}{dd_1^\alpha}\right)P_t\{\beta_{(\nu_t)} > d \wedge d^{1/c\alpha}\}\\ &+ \frac{2}{d_1^\alpha}E_t(\beta_{(\nu_t)}^{\alpha(1-c)}) + e^{-t}\sup_{\xi\in\mathbb{R}}|Im(v_0(\xi))|\end{aligned}$$

is valid for any c in $(0, 1)$.

The upper bound provided in Theorem 2.2 goes to zero as $t \to +\infty$ thanks to (F_1), (F_4) and the definition of $\bar{v}_0^*$. Then, it can be used to yield further bounds, either via the statement of specific upper bounds for the expectations which appear in the right-hand side or through the adoption of suitable extra conditions on v_0. As to the former way of arguing, it is worth recalling that Proposition 8 in [18] gives

$$\begin{aligned}E_t\left(\sum_{j=1}^{\nu_t}|\beta_{j,t}|^m\right) &= E_t\left(\sum_{j=1}^{\nu_t}A_{m(1+p)}^{\delta_j}\right) \quad (\delta_j = \text{ depth of leaf } j)\\ &= \exp\{-t(1 - 2A_{m(1+p)})\} \quad (m \geq 0)\end{aligned} \tag{19}$$

with A_m defined as in (11). Moreover, from Lemma 1 in [19],

$$P_t\{\beta_{(\nu_t)} > x\} \leq x^{-\frac{q}{1+p}}e^{-t(1-2A_q)} \quad (0 < x < 1, q > 0) \tag{20}$$

which, in turn, yields

$$E_t(\beta_{(\nu_t)}^m) \leq e^{-\sigma m t} + e^{-t(1-q\sigma\alpha/2-2A_q)} \tag{21}$$

for any positive σ and q. Now, define $\mathcal{U}_{1,t}$ as

$$\mathcal{U}_{1,t} := \bar{v}_0^*(d_1\beta^c_{(\nu_t)}) + 2M_1(\bar{v}_0^*(d_1\beta^c_{(\nu_t)}))^2 + \frac{4}{5}M^2M_1\beta^{\alpha}_{(\nu_t)} + \frac{32}{25}M_3M^4\beta^{2\alpha}_{(\nu_t)}$$
$$+ \left(k^* + \frac{2}{dd_1^{\alpha}}\right)\mathbb{I}\{\beta_{(\nu_t)} > d \wedge d^{1/c\alpha}\} + \frac{2}{d_1^{\alpha}}\beta^{\alpha(1-c)}_{(\nu_t)} + e^{-t}\sup_{\xi\in\mathbb{R}}|Im(v_0(\xi))|$$

and set

$$\mathcal{M}_{1,t} := \bar{v}_0^*(d_1\beta^c_{(\nu_t)}) + 2M_1(\bar{v}_0^*(d_1\beta^c_{(\nu_t)}))^2$$
$$\mathcal{R}_{1,t} := \mathcal{U}_{1,t} - \mathcal{M}_{1,t}.$$

Next, observe that the upper bound provided by Theorem 2.2 can be written as

$$E_t(\mathcal{M}_{1,t}) + E_t(\mathcal{R}_{1,t}) \le E_t(\mathcal{M}_{1,t};\beta_{(\nu_t)} \le x_t) + M(1+2M_1M)P_t\{\beta_{(\nu_t)} > x_t\} + E_t(\mathcal{R}_{1,t})$$

with $x_t := \exp\{-\sigma t\}$ and σ satisfying $1 - 2A_q - \sigma q/(1+p) > 0$ to obtain

$$\chi_\alpha(F(\cdot,t),G_\alpha) \le \bar{v}_0^*(d_1e^{-c\sigma t}) + 2M_1\bar{v}_0^*(d_1e^{-c\sigma t})^2$$
$$+ M(1+2M_1M)e^{-t(1-2A_q-\sigma q/(1+p))} + E_t(\mathcal{R}_{1,t}). \tag{22}$$

Then, since $E_t(\mathcal{R}_{1,t})$ can be re-written—thanks to (20)–(21)—as a sum of exponential functions, (22) provides a bound entirely expressed, through $\bar{v}_0^*$, in terms of exponential functions of t.

Exponential rates of relaxation to equilibrium hold true under some extra condition concerning the local behavior of v_0 near the origin.

Theorem 2.3 *Assume that, in addition to the assumptions made in* Theorem 2.2, (12) *holds for some* $\delta > 0$. *Moreover, let* d *be chosen in such a way that* $|x| \le d_1d^{1/\alpha}$ *entails* $|v_0(x)| \le \rho|x|^\delta$ *for some* $\rho > 0$. *Then*,

$$\chi_{\alpha+\delta}\big(F(\cdot,t),G_\alpha\big) \le \left(\rho + \frac{2}{d_1^{\alpha+\delta}d^{(\alpha+\delta)/\alpha}}\right)e^{-t(1-2A_{2(1+\delta/\alpha)})}$$
$$+ \frac{4}{5}M^2M_{\frac{\alpha-\delta}{\alpha}}e^{-t(1-2A_4)} + 2\rho^2M_{\frac{\alpha+\delta}{\alpha}}e^{-t(1-2A_{2(1+2\delta/\alpha)})}$$
$$+ \frac{32}{25}M^4M_{\frac{3\alpha-\delta}{\alpha}}e^{-t(1-2A_6)} + e^{-t}\sup_{\xi\in\mathbb{R}}\frac{1}{|\xi|^\delta}|Im(v_0(\xi))|$$

holds true for δ *in* $(0,\alpha]$, *while*

$$\chi_{2\alpha}\big(F(\cdot,t),G_\alpha\big) \le \left(\frac{4}{5}M^2 + \frac{2}{d_1^{2\alpha}d^2}\right)e^{-t(1-2A_4)}$$
$$+ \rho M_{\frac{\delta-\alpha}{\alpha}}e^{-t(1-2A_{2(1+\delta/\alpha)})} + \frac{32}{25}M^4M_2e^{-t(1-2A_6)}$$
$$+ 2\rho^2M_{\frac{2\delta}{\alpha}}e^{-t(1-2A_{2(1+2\delta/\alpha)})} + e^{-t}\sup_{\xi\in\mathbb{R}}\frac{1}{|\xi|^\alpha}|Im(v_0(\xi))|$$

is verified for δ *in* $(\alpha, 2\alpha]$.

In short, this proposition can be condensed into the following statement: *Under the hypotheses of* Theorem 2.3, *there are constants a_1 and a_2 such that*:

$$\chi_{\alpha+\delta}(F(\cdot,t),G_\alpha) \le a_1 e^{-t(1-2A_{2(1+\delta/\alpha)})} \quad \text{if } \delta \in (0,\alpha],$$
$$\chi_{2\alpha}(F(\cdot,t),G_\alpha) \le a_2 e^{-t(1-2A_4)} \quad \text{if } \delta \in (\alpha,2\alpha].$$

Statements of the same type as Theorems 2.2 and 2.3 are proved in Sect. 5 in [20] for $\alpha = 2$ $(p = 0)$, i.e. when G_α is a Gaussian distribution function with zero mean. Notice that the rate of convergence given in the former part of the last theorem coincides with that of Toscani and Pulvirenti previously quoted in (F_5). The latter part of Theorem 2.3 and, mainly, Theorem 2.2 seem to be new. See Sect. 2.4 for further comments.

2.2 Rates of Relaxation to Equilibrium in Kolmogorov's Metric (Conditions Expressed on the Characteristic Function ϕ_0)

Rates of convergence of $F(\cdot,t)$ to G_α, in Kolmogorov's metric, can be deduced from the representation of V_t as weighted sum, via the well-known Berry-Esseen inequality in its form given, for example, in Theorem 3.18 of [22]. It is worth recalling that application of this inequality is allowed thanks to the fact that G_α has derivatives of all orders at every point. Henceforth, given any strictly positive l and q, we put

$$N_l = \int_0^{+\infty} \exp\{-(a_0-\eta)\xi^\alpha\}\xi^{l-1} d\xi$$

and

$$H(\xi,q) := |v_0^*(\xi q)|(1+2|\xi|^\alpha |v_0^*(\xi q)|), \qquad \bar{H}(\xi,q) := \sup_{u\le q} H(\xi,u)$$

with v_0^* as in (14).

Theorem 2.4 *If F_0^* belongs to the domain of normal attraction of G_α with $\alpha = 2/(1+p)$ for some $p > 0$, then*

$$\begin{aligned} K(F(\cdot,t),G_\alpha) \le{}& \frac{2}{\pi} E_t\left[\sum_{j=1}^{\nu_t} |\beta_{j,t}|^\alpha \int_0^{+\infty} H(\xi,|\beta_{j,t}|)\xi^{\alpha-1} e^{-(a_0-\eta)\xi^\alpha} d\xi\right] \\ &+ \frac{\mathbf{c}\|g_\alpha\|}{\tilde{d}} E_t(\beta_{(\nu_t)}) + \frac{8}{5\pi} M^2 N_{2\alpha} e^{-t(1-2A_4)} + \frac{64}{25\pi} M^4 N_{4\alpha} e^{-t(1-2A_6)} \\ &+ \frac{e^{-t}}{2} \sup_{x\in\mathbb{R}} |F_0(x)+F_0(-x-0)-1| \end{aligned}$$

$\mathbf{c}$ *being the constant which appears in the above-mentioned version of the Berry-Esseen inequality and $\tilde{d} := (3d/8M)^{1/\alpha}$.*

A further bound for $K(F(\cdot,t),G_\alpha)$ can be obtained by replacing the summand

$$\frac{2}{\pi} E_t\left[\sum_{j=1}^{\nu_t} |\beta_{j,t}|^\alpha \int_0^{+\infty} H(\xi,|\beta_{j,t}|)\xi^{\alpha-1} e^{-(a_0-\eta)\xi^\alpha} d\xi\right]$$

with

$$\frac{2}{\pi} E_t \left[\int_0^{+\infty} \bar{H}(\xi, \beta_{(\nu_t)}) \xi^{\alpha-1} e^{-(a_0-\eta)\xi^\alpha} d\xi \right].$$

Finally, it is worth presenting a bound of the same style as (22), entirely depending on exponential functions:

$$\begin{aligned} K(F(\cdot,t), G_\alpha) \le{}& \frac{8}{5\pi} M^2 N_{2\alpha} e^{-t(1-2A_4)} + \frac{64}{25\pi} M^4 N_{4\alpha} e^{-t(1-2A_6)} \\ &+ \frac{e^{-t}}{2} \sup_{x\in\mathbb{R}} |F_0(x) + F_0(-x-0) - 1| \\ &+ \left(\frac{2}{\pi} \|v_0^*\| (N_\alpha + 2N_{2\alpha} \|v_0^*\|) + \frac{\mathbf{c}\|g_\alpha\|}{\tilde{d}} \right) e^{-t(1-q\sigma\alpha/2-2A_q)} \\ &+ e^{-\rho t} \frac{\mathbf{c}\|g_\alpha\|}{\tilde{d}} + \frac{2}{\pi} \int_0^{+\infty} \bar{H}(\xi, e^{-t\sigma}) \xi^{\alpha-1} e^{-(a_0-\eta)\xi^\alpha} d\xi. \end{aligned}$$

Notice that the above two bounds go to zero as $t \to +\infty$. Indeed, the latter goes to zero since, on the one hand, $\lim_{t\to+\infty} \int_0^{+\infty} \bar{H}(\xi, e^{-t\sigma}) \xi^{\alpha-1} e^{-(a_0-\eta)\xi^\alpha} d\xi = 0$ and, on the other hand, σ and q can be chosen in such a way that $1 - q\sigma\alpha/2 - 2A_q$ turns out to be strictly positive. Exponential bounds can be given under the usual condition on the behavior of v_0 near the origin.

Theorem 2.5 *If, besides the assumptions considered in* Theorem 2.4, v_0^* *is such that* $|v_0^*(\xi)| = O(|\xi|^\delta)$ *as* $\xi \to 0$ *for some* $\delta > 0$, *and* d *is chosen to make sure that* $|\xi| \le \tilde{d} = (3d/8m)^{1/\alpha}$ *entails* $|v_0^*(\xi)| \le \rho|\xi|^\delta$, *then*

$$\begin{aligned} K(F(\cdot,t), G_\alpha) \le{}& \frac{8}{5\pi} M^2 N_{2\alpha} e^{-t(1-2A_4)} + \frac{64}{25\pi} M^4 N_{4\alpha} e^{-t(1-2A_6)} \\ &+ \frac{2}{\pi} \rho N_{\alpha+\delta} e^{-t(1-2A_{2+2\delta/\alpha})} + 2\rho^2 N_{2\alpha+2\delta} e^{-t(1-2A_{2+4\delta/\alpha})} \\ &+ \frac{\mathbf{c}\|g_\alpha\|}{\tilde{d}} E_t(\beta_{(\nu_t)}) + \frac{e^{-t}}{2} \sup_{x\in\mathbb{R}} |F_0(x) + F_0(-x-0) - 1|. \end{aligned}$$

In view of (21), the thesis of Theorem 2.5 can be formulated as: *There are positive constants* a_3 *and* b *such that* $K(F(\cdot,t), G_\alpha) \le a_3 e^{-bt}$ *for every* $t \ge 0$.

2.3 Convergence in Kolmogorov's Metric (Conditions Expressed on the Initial Probability Distribution F_0)

A characteristic feature of the results presented until now is that all the assumptions adopted to obtain bounds—in particular, extra conditions to achieve exponentially fast convergence—are formulated in terms of conditions on the initial characteristic function. In general, with respect to the actual choice of initial data, it is easier and more natural to assign conditions on F_0 than on ϕ_0. Apropos of this remark, see the role played by Lemma 6.1 in [30] and in Sect. 2.4 below. With reference to the classical case of independent and identically distributed summands, see, for example, [12, 13, 24]. Accordingly, the main objective of the rest of the section is to determine bounds for $K(F(\cdot,t), G_\alpha)$, expressed in terms of quantities whose computation is generally easier than the computation of characteristic functions, once either F_0 or some approximate form of F_0 has been assigned. To pave the way for presentation, let us complement previous notation given, in particular, in Sect. 1.2:

$$h^*(x) := x^\alpha S^*(x) = x^\alpha\{1 - F_0^*(x)\} - c_0^* = x^\alpha F_0^*(-x) - c_0^* \quad (x > 0)$$

$$b_1^*(x) := 2x \int_D^{+\infty} \sin(xu) S^*(u) du$$

where D is some strictly positive number and the integral has to be meant as improper Riemann integral. Moreover,

$$B_1 := 2k_1 N_2 + 8k_1 k_2 N_{2+\alpha}, \qquad B_2 := 8k_1^2 N_4, \qquad B_3 := 4k_2 N_{1+\alpha} + 2N_1$$

$$B_4 := 4k_2 N_{2+\alpha} + 2N_2, \qquad B_5 := \frac{4}{5} M^2 N_{2\alpha}, \qquad B_6 := \frac{32}{25} M^4 N_{4\alpha}$$

with

$$k_1 := \int_0^D x|S^*(x)|dx, \qquad k_2 := \sup_{x>0} \frac{|b_1^*(x)|}{x^\alpha} \le \max\left\{\|v_0^*\| + 2k_1, 2\int_D^{+\infty} |S^*(x)|dx\right\}$$

and

$$H_1^*(q) := \int_0^1 y^{1-\alpha}|h^*(y/q)|dy, \qquad H_2^*(q) := \int_1^{+\infty} y^{-\alpha}|h^*(y/q)|dy$$

$$k_3 := \sup_{q\in(0,1)} H_1^*(q), \qquad k_4 := \sup_{q\in(0,1)} H_2^*(q).$$

Theorem 2.6 *If F_0^* belongs to the domain of normal attraction of G_α with $\alpha = 2/(1+p)$ in $[1, 2)$, and $\int_{\mathbb{R}} |S^*(x)|dx < +\infty$ if $\alpha = 1$, then*

$$\begin{aligned} K(F(\cdot,t), G_\alpha) \le{}& \frac{2}{\pi} E_t\left[\sum_{j=1}^{\nu_t} |\beta_{j,t}|^\alpha \{B_3 H_1^*(|\beta_{j,t}|) + B_4 H_2^*(|\beta_{j,t}|)\}\right] \\ &+ \frac{\mathbf{c}\|g_\alpha\|}{\tilde{d}} E_t(\beta_{(\nu_t)}) + \frac{2}{\pi}\Big\{B_1 e^{-t(1-2A_{4/\alpha})} + B_2 e^{-t(1-2A_{(8-2\alpha)/\alpha})} \\ &+ B_5 e^{-t(1-2A_4)} + B_6 e^{-t(1-2A_6)}\Big\} + \frac{e^{-t}}{2} \sup_{x\in\mathbb{R}} |F_0(x) + F_0(-x-0) - 1|. \end{aligned}$$

Then, setting $\bar{H}_i^*(x) := \sup_{y\le x} H_i^*(y)$ for $i = 1, 2$, and recalling (21), we obtain a more expressive form for the bound, that is

$$\begin{aligned} K(F(\cdot,t), G_\alpha) \le{}& \frac{2}{\pi}\Big\{B_1 e^{-t(1-2A_{4/\alpha})} + B_2 e^{-t(1-2A_{(8-2\alpha)/\alpha})} \\ &+ \left(k_3 B_3 + k_4 B_4 + \frac{\pi}{2}\frac{\mathbf{c}\|g_\alpha\|}{\tilde{d}}\right) e^{-t(1-q\sigma\alpha/2-2A_q)} \\ &+ B_5 e^{-t(1-2A_4)} + B_6 e^{-t(1-2A_6)} + \frac{\pi}{2}\frac{\mathbf{c}\|g_\alpha\|}{\tilde{d}} e^{-\sigma t} \\ &+ B_3 \bar{H}_1^*(e^{-\sigma t}) + B_4 \bar{H}_2^*(e^{-\sigma t})\Big\} + \frac{e^{-t}}{2} \sup_{x\in\mathbb{R}} |F_0(x) + F_0(-x-0) - 1|. \end{aligned}$$

In order to obtain exponential bounds, we reinforce the assumptions made in Theorem 2.6, in the sense that

$$|h^*(x)| \le \frac{\rho'}{|x|^\delta} \quad \text{for some positive constant } \rho' \text{ and } \delta \text{ in } (0, 2-\alpha). \tag{23}$$

Theorem 2.7 *Besides the assumptions made in* Theorem 2.6, *suppose* (23) *holds true. Then,*

$$\begin{aligned} K(F(\cdot,t), G_\alpha) \le \frac{2}{\pi}\Big\{ & B_1 e^{-t(1-2A_{4/\alpha})} + B_2 e^{-t(1-2A_{(8-2\alpha)/\alpha})} + B_5 e^{-t(1-2A_4)} \\ & + B_6 e^{-t(1-2A_6)} + \left(\frac{\rho' B_3}{2-\alpha-\delta} + \frac{\rho' B_4}{\alpha+\delta-1}\right) e^{-t(1-2A_{2+2\delta/\alpha})}\Big\} \\ & + \frac{\mathbf{c}\|g_\alpha\|}{\tilde{d}}(e^{-\sigma t} + e^{-t(1-q\sigma\alpha/2-2A_q)}) + \frac{e^{-t}}{2}\sup_{x\in\mathbb{R}}|F_0(x) + F_0(-x-0) - 1| \end{aligned}$$

which is tantamount to saying that there are positive constants a_4 and b_4 such that $K(F(\cdot,t), G_\alpha) \le a_4 e^{-b_4 t}$ holds for every $t \ge 0$.

It remains to consider the case with α in $(0,1)$. In point of fact, the next theorem is valid for any α in $(0,2)$, but it requires further notation. Firstly, S^* is assumed to be *monotonic* on $[D, +\infty)$. Then, one sets

$$b_2^*(x) := -2\int_D^{+\infty}(1-\cos(xy))dS^*(y);$$

$$H_3^*(q) := \int_1^{+\infty} y^{-(1+\alpha)}|h^*(y/q)|dy, \qquad \bar{H}_3^*(q) := \sup_{y\le q} H_3^*(y), \qquad k_5 := \sup_{q\in(0,1)} H_3^*(q);$$

$$\bar{B}_1 := 2\bar{k}_1 N_2 + 8\bar{k}_1\bar{k}_2 N_{2+\alpha} + |S^*(D)|D^2 N_2 + 2\bar{k}_2|S^*(D)|D^2 N_{2+\alpha},$$

$$\bar{B}_2 := 8\bar{k}_1^2 N_4, \qquad \bar{B}_3 := 2z_0 + 4z_\alpha\bar{k}_2$$

with

$$\begin{aligned} \bar{k}_1 &:= k_1 + \frac{D^2|S^*(D)|}{2}, \\ \bar{k}_2 &:= \sup_{x>0}\frac{|b_2^*(x)|}{x^\alpha} \le k_2 + 2D|S^*(D)|\max\left(\frac{D^2}{2}, 2\right), \\ z_r &:= \max\left\{\int_0^{+\infty}\left|\frac{d}{dx}n_r(x)\right|dx, \frac{1}{2}\int_0^{+\infty}x^2\left|\frac{d}{dx}n_r(x)\right|dx\right\} \end{aligned}$$

where

$$n_r(x) := e^{-(a_0-\eta)x^\alpha}x^r \quad x > 0.$$

Theorem 2.8 *Let α belong to $(0,2)$ and let S^* be monotonic on $[D,+\infty)$. Then,*

$$K(F(\cdot,t), G_\alpha) \le \frac{2}{\pi}\bar{B}_3 E_t\left[\sum_{j=1}^{\nu_t}|\beta_{j,t}|^\alpha\{H_1^*(|\beta_{j,t}|) + H_3^*(|\beta_{j,t}|)\}\right]$$

$$+\frac{\mathbf{c}\|g_\alpha\|}{\tilde{d}}E_t(\beta_{(\nu_t)})+\frac{2}{\pi}\Big\{\bar{B}_1e^{-t(1-2A_{4/\alpha})}+\bar{B}_2e^{-t(1-2A_{(8-2\alpha)/\alpha})}$$
$$+B_5e^{-t(1-2A_4)}+B_6e^{-t(1-2A_6)}\Big\}+\frac{e^{-t}}{2}\sup_{x\in\mathbb{R}}|F_0(x)+F_0(-x-0)-1|.$$

As done elsewhere in this section, it should be noted that the inequality

$$\frac{\mathbf{c}\|g_\alpha\|}{\tilde{d}}E_t(\beta_{(\nu_t)})+\frac{2}{\pi}\bar{B}_3E_t\Big[\sum_{j=1}^{\nu_t}|\beta_{j,t}|^\alpha\{H_1^*(|\beta_{j,t}|)+H_3^*(|\beta_{j,t}|)\}\Big]$$
$$\le\Big(\frac{\mathbf{c}\|g_\alpha\|}{\tilde{d}}+\frac{2}{\pi}\bar{B}_3(k_3+k_5)\Big)e^{-t(1-q\sigma\alpha/2-2A_q)}$$
$$+\frac{\mathbf{c}\|g_\alpha\|}{\tilde{d}}e^{-\sigma t}+\frac{2}{\pi}\bar{B}_3\{\bar{H}_1^*(e^{-\sigma t})+\bar{H}_3^*(e^{-\sigma t})\}$$

is useful to yield a bound for $K(F(\cdot,t),G_\alpha)$ depending only on exponential functions, while an exponential bound can be derived from the next theorem.

Theorem 2.9 *Besides the assumptions made in* Theorem 2.8, *suppose* (23) *holds true. Then*,

$$K(F(\cdot,t),G_\alpha)\le\frac{2}{\pi}\Big\{\bar{B}_1e^{-t(1-2A_{4/\alpha})}+\bar{B}_2e^{-t(1-2A_{(8-2\alpha)/\alpha})}+B_5e^{-t(1-2A_4)}$$
$$+B_6e^{-t(1-2A_6)}+\Big(\frac{\rho'\bar{B}_3}{2-\alpha-\delta}+\frac{\rho'\bar{B}_3}{\alpha+\delta}\Big)e^{-t(1-2A_{2+2\delta/\alpha})}\Big\}$$
$$+\frac{\mathbf{c}\|g_\alpha\|}{\tilde{d}}(e^{-\sigma t}+e^{-t(1-q\sigma\alpha/2-2A_q)})+\frac{e^{-t}}{2}\sup_{x\in\mathbb{R}}|F_0(x)+F_0(-x-0)-1|.$$

2.4 Brief Comparative Study of Extra Condition on ϕ_0 and on F_0

In view of the greater expressiveness of assumptions given for F_0, if compared to conditions on ϕ_0, already stressed at the beginning of Sect. 2.3, we conclude the section with a brief comparative analysis. This analysis deals, on the one hand, with the two kinds of conditions actually used in the present paper and, on the other hand, with our conditions on F_0 compared with those introduced in [30].

Recall that in Sects. 2.1 and 2.2 we used an extra condition which, in the symmetric case, reduces to

$$|v_0^*(\xi)|=O(|\xi|^\delta)\quad\text{as }\xi\to 0,\ \text{for some }\delta>0\tag{24}$$

while, in Sect. 2.3, we have stated a few results under the extra condition

$$\Big|(1-F_0^*(x))-\frac{c_0^*}{x^\alpha}\Big|\le\frac{\rho'}{x^{\alpha+\delta}}\quad(x>0)\tag{25}$$

for some δ in $(0,2-\alpha)$ when α belongs to $[1,2)$, and for some δ in $(0,2-\alpha)$ when α belongs to $(0,1)$ provided that $S^*(x)=(1-F_0^*(x))-c_0^*x^{-\alpha}$ is monotonic for $x>D\ge 0$.

As to the former point under discussion, notice that for α in $[1,2)$ one can resort to easy inequalities, to be explained and used in the proof of Proposition 3.6, to obtain

$$|v_0^*(\xi)|\le\frac{|b_1^*(\xi)|}{|\xi|^\alpha}+2k_1|\xi|^{2-\alpha}$$

where, in view of (25), $|b_1^*(\xi)| = O(|\xi|^{\alpha+\delta})$. An analogous conclusion holds true when $0 < \alpha < 1$ with b_2^* and $\bar{k}_1$ in the place of b_1^* and k_1, respectively. See formal developments in the proof of Proposition 3.7. Hence: *If* δ *belongs to* $(0, 2-\alpha)$ *with* $0 < \alpha < 2$, *and* S^* *is monotonic on* $(D, +\infty)$ *for some* $D \geq 0$ *when* $0 < \alpha < 1$, *then* (25) *entails* (24).

Moving on to the latter kind of comparisons, it should be recalled that [30], in order that initial data can satisfy (25), assume

$$m_{\alpha+\delta} := \int_{\mathbb{R}} |x|^{\alpha+\delta} |f_0(x) - g_\alpha(x)| dx < +\infty \quad \text{for some } \delta > 0. \tag{26}$$

In Sect. 4 of [23] it is proved that (26) *entails* (24) and now we prove that (26) *yields* (25) *when* $\delta \leq \alpha$. Indeed, from the Markov inequality,

$$|F^*(x) - G_\alpha(x)| \leq \frac{m_{\alpha+\delta}}{2x^{\alpha+\delta}}.$$

This, combined with a well-known asymptotic expression for G_α (see, for example, Sects. 2.4 and 2.5 of [34]), gives

$$\left| F^*(x) - \frac{c_0^*}{x^\alpha} \right| \leq \frac{m_{\alpha+\delta}}{2x^{\alpha+\delta}} + O\left(\frac{1}{x^{2\alpha}}\right) \quad (x \to +\infty).$$

Then, (25) *follows from* (26) *when* $\delta \leq \alpha$. This last restriction is consistent with Theorem 6.2 in [30], mentioned in (F_5), and with the first part of Theorem 2.3. Moreover, it should be noted that classical asymptotic formulae for g_α (see, e.g., [26]) can be applied to exhibit simple examples of initial data which meet (25) but do not meet (26). In other words, the criterion evoked by [30]—to get (24) together with exponential bounds for $\chi_{\alpha+\delta}$ with $\delta \leq \alpha$—could be usefully replaced by the weaker condition (25), as we have done for convergence with respect to the Kolmogorov metric.

3 Limit Theorems for Weighted Sums of Independent Random Numbers

As mentioned in the introductory paragraph of Sect. 2—see, in particular, explanation for (16), (17) and (18)—the present section focuses on the study of the convergence in distribution of weighted sums of independent random variables. This study, besides the interest it could hold in itself, is essential to prove the theorems already stated in Sect. 2. In point of fact, the main steps of the arguments used to prove these theorems are set out in the propositions we are about to enunciate and prove in the present section. Specific indications of how they are used will be given in the Appendix.

For the present, it should be recalled that we are interested in convergence in distribution of sums

$$S_n := \sum_{j=1}^{n} q_j^{(n)} X_j \tag{27}$$

with $X_1, X_2, \ldots$ *independent and identically distributed real-valued random variables with common distribution function* F_0. Moreover, the numbers $q_j^{(n)}$ *are assumed to satisfy* (17)–(18), and F_0 is supposed to be a *symmetric element of the domain of normal attraction of* (7). Then according to (F_3) and (F_4), there is $c_0 \geq 0$ satisfying

$$a_0 = 2c_0 \int_0^{+\infty} \frac{\sin(x)}{x^\alpha} dx \tag{28}$$

for which

$$\lim_{x\to-\infty}|x|^\alpha F_0(x) = \lim_{x\to+\infty} x^\alpha\{1-F_0(x)\} = c_0$$

and

$$1-\phi_0(\xi) = (a_0 + v_0(\xi))|\xi|^\alpha \quad (\xi \in \mathbb{R})$$

where v_0 is a bounded real-valued function satisfying $|v_0(\xi)| = o(1)$ as $\xi \to 0$. See (13)–(14).

The above conditions, in italics, are assumed to be in force throughout the present section, and will not be repeated in the following statements. It is worth recalling that these statements are inspired by previous work published in [12, 13, 24]. Accordingly, the present line of reasoning is based on certain inequalities contained in the following lemma where, as in the rest of the section, for the sake of typographic convenience, q_j is used instead of $q_j^{(n)}$.

Lemma 3.1 *Let $\tilde\phi_n$ be the characteristic function of* (27). *Then,*

$$\begin{aligned}
|\tilde\phi_n(\xi) - \hat g_\alpha(\xi)| \le e^{-(a_0-\eta)|\xi|^\alpha}|\xi|^\alpha &\Bigg\{\sum_{j=1}^n q_j^\alpha |v_0(\xi q_j)|(1+2|\xi|^\alpha|v_0(\xi q_j)|) \\
&+ |\xi|^\alpha M^2 \sum_{j=1}^n q_j^\alpha\left(\frac{4}{5}q_j^\alpha + \frac{32}{25}M^2|\xi|^{2\alpha}q_j^{2\alpha}\right)\Bigg\}\mathbb{I}\{|\xi|\le D_n\} \\
&+ 2\mathbb{I}\{|\xi| > D_n\}\left(\frac{|\xi|}{d_1}\right)^s\left[\frac{q_{(n)}^s}{d^{s/\alpha}}\mathbb{I}\{c=0\} + \frac{q_{(n)}^s}{d^{s/\alpha}}\mathbb{I}\{q_{(n)} > d^{1/c\alpha},\ 0<c<1\}\right. \\
&\left.+ q_{(n)}^{s(1-c)}\mathbb{I}\{q_{(n)} \le d^{1/c\alpha}, 0<c<1\}\right] \qquad (29)
\end{aligned}$$

holds for any ξ in $\mathbb{R}$, $s>0$, c in $[0,1)$, d, d_1, k^* and M being the same as in Theorem 2.2 *with v_0 in the place of v_0^*, $q_{(n)} = max\{q_1,\dots q_n\}$ and $D_n = D_n(c,d) := (\frac{3}{8M}(d\wedge q_{(n)}^{c\alpha}))^{1/\alpha}q_{(n)}^{-1}\mathbb{I}\{0<c<1\} + (\frac{3}{8M}d)^{1/\alpha}q_{(n)}^{-1}\mathbb{I}\{c=0\}$. Moreover, for $s=\alpha$, c in $(0,1)$ and ξ in $\mathbb{R}$,*

$$\begin{aligned}
|\tilde\phi_n(\xi) - \hat g_\alpha(\xi)| \le |\xi|^\alpha\Bigg[&e^{-(a_0-\eta)|\xi|^\alpha}\Bigg(k^*\mathbb{I}\Big\{q_{(n)} > d,\ |\xi| \le \frac{d_1 d^{1/\alpha}}{q_{(n)}}\Big\} \\
&+ \bar\sigma(\xi) q_{(n)}^\alpha \mathbb{I}\Big\{q_{(n)} \le d^{1/c\alpha}, |\xi| \le d_1 q_{(n)}^{c-1}\Big\}\Bigg) \\
&+ \frac{2}{d_1^\alpha}\left(\frac{q_{(n)}^\alpha}{d}\mathbb{I}\{q_{(n)} > d^{1/c\alpha}\} + q_{(n)}^{\alpha(c-1)}\right)\Bigg] \qquad (30)
\end{aligned}$$

with

$$\bar\sigma(\xi) = \sum_{j=1}^n q_j^\alpha|v_0(\xi q_j)| + |\xi|^\alpha\left(\frac{4}{5}M^2\sum_{j=1}^n q_j^{2\alpha} + 2\sum_{j=1}^n q_j^\alpha|v_0(\xi q_j)|^2 + \frac{32}{25}|\xi|^{3\alpha}M^4\sum_{j=1}^n q_j^{3\alpha}\right).$$

Proof According to previous notation, set $\|v_0\| := \sup_{\{x>0\}} |v_0(x)|$ and $\bar{v}_0(\xi) := \sup_{\{0<x\le\xi\}} |v_0(x)|$. Now, in view of (F_4),

$$|1-\phi_0(\xi q_j)| = |a_0 + v_0(\xi q_j)||\xi q_j|^\alpha \le M|\xi|^\alpha q_j^\alpha$$

and the last term turns out to be bounded from above by $3d/8 \le 3/8$ when $|\xi|q_{(n)} \le (3d/8M)^{1/\alpha}$. Since $\log(1+z) = z + (4/5)\theta_z|z|^2$ for $|z| \le 3/8$ and some θ_z satisfying $|\theta_z| \le 1$ (see, for example, Lemma 3 in Sect. 9.1 of [11]), then $|\xi|q_{(n)} \le (3d/8M)^{1/\alpha}$ yields

$$\begin{aligned}\tilde{\phi}_n(\xi) &= \exp\left\{\sum_{j=1}^n \log(\phi_0(\xi q_j))\right\} = \exp\left\{\sum_{j=1}^n \log(1-(1-\phi_0(\xi q_j)))\right\} \\ &= \exp\left\{-\sum_{j=1}^n (1-\phi_0(\xi q_j)) + \sum_{j=1}^n r(1-\phi_0(\xi q_j))\right\}\end{aligned}$$

with $r(x) := (4\theta_{-x}/5)|x|^2$. Moreover, if $|\xi| \le (3d/8M)^{1/\alpha}/q_{(n)}$ and $0<d<1$,

$$|r(1-\phi_0(\xi q_j))| \le \frac{4}{5}M^2|\xi|^{2\alpha}q_j^{2\alpha} \quad (j=1,\dots,n)$$

and, via (F_4),

$$\begin{aligned}\tilde{\phi}_n(\xi) &= \exp\Big(-\sum_{j=1}^n \{a_0 + v_0(\xi q_j)\}|\xi q_j|^\alpha + \sum_{j=1}^n r(1-\phi_0(\xi q_j))\Big) \\ &= \exp(-a_0|\xi|^\alpha)\exp(-B_n(\xi) + R_{1,n}(\xi))\end{aligned} \tag{31}$$

with

$$B_n(\xi) = |\xi|^\alpha \sum_{j=1}^n q_j^\alpha v_0(\xi q_j)$$

and

$$|R_{1,n}(\xi)| = \left|\sum_{j=1}^n r(1-\phi_0(\xi q_j))\right| \le \frac{4}{5}M^2|\xi|^{2\alpha}\sum_{j=1}^n q_j^{2\alpha}.$$

Writing

$$\begin{aligned}\exp(-B_n(\xi)+R_{1,n}(\xi)) &= 1 - B_n(\xi) + R_{1,n}(\xi) \\ &\quad + (R_{1,n}(\xi) - B_n(\xi))^2 \sum_{l\ge 0} \frac{(R_{1,n}(\xi)-B_n(\xi))^l}{l!}\frac{l!}{(l+2)!} \\ &= 1 - B_n(\xi) + R_{1,n}(\xi) + R_{2,n}(\xi),\end{aligned}$$

with

$$\begin{aligned}|R_{2,n}(\xi)| &= (R_{1,n}(\xi) - B_n(\xi))^2\Big|\sum_{l\ge 0}\frac{(R_{1,n}(\xi)-B_n(\xi))^l}{l!}\frac{l!}{(l+2)!}\Big| \\ &\le 2\{B_n(\xi)^2 + R_{1,n}(\xi)^2\}\exp(|B_n(\xi)| + |R_{1,n}(\xi)|),\end{aligned} \tag{32}$$

equalities (31) give

$$\tilde{\phi}_n(\xi) = \exp(-a_0|\xi|^\alpha)\{1 - B_n(\xi) + R_{1,n}(\xi) + R_{2,n}(\xi)\}. \tag{33}$$

As to $R_{2,n}(\xi)$, for $|\xi|^\alpha \le (3d/8M)q_{(n)}^{-\alpha}$ and any sufficiently small d, one gets

$$\begin{aligned}|B_n(\xi)| + |R_{1,n}(\xi)| &\le |\xi|^\alpha \sum_{j=1}^n \bar{v}_0(\xi q_{(n)})q_j^\alpha + \frac{4}{5}M^2|\xi|^{2\alpha}q_{(n)}^\alpha \sum_{j=1}^n q_j^\alpha \\ &\le |\xi|^\alpha \left\{\bar{v}_0(\xi q_{(n)}) + \frac{4}{5}M^2|\xi|^\alpha q_{(n)}^\alpha\right\} \le \eta|\xi|^\alpha\end{aligned}$$

by the definition of d given immediately before the beginning of Sect. 2.1. This entails

$$\exp(|B_n(\xi)| + |R_{1,n}(\xi)|) \le e^{\eta|\xi|^\alpha}$$

for any η in $(0, a_0)$ and $|\xi| \le (3d/8M)^{1/\alpha}q_{(n)}^{-1}$. Next, an application of Jensen's inequality yields

$$|B_n(\xi)|^2 + |R_{1,n}(\xi)|^2 \le |\xi|^{2\alpha}\sum_{j=1}^n q_j^\alpha v_0(\xi q_j)^2 + \frac{16}{25}M^4|\xi|^{4\alpha}\sum_{j=1}^n q_j^{3\alpha}$$

which, in turn, combined with (32), gives

$$|R_{2,n}(\xi)| \le \left\{2|\xi|^{2\alpha}\sum_{j=1}^n q_j^\alpha v_0(\xi q_j)^2 + \frac{32}{25}M^4|\xi|^{4\alpha}\sum_{j=1}^n q_j^{3\alpha}\right\}e^{\eta|\xi|^\alpha}.$$

Now, from (33) with $|\xi| \le D_n$,

$$\begin{aligned}|\tilde{\phi}_n(\xi) - \exp(-a_0|\xi|^\alpha)| &\le e^{-a_0|\xi|^\alpha}|\xi|^\alpha\left\{\sum_{j=1}^n |v_0(\xi q_{(n)})|q_j^\alpha + \frac{4}{5}M^2|\xi|^\alpha\sum_{j=1}^n q_j^{2\alpha}\right. \\ &\quad \left. + \left(2|\xi|^\alpha\sum_{j=1}^n q_j^\alpha v_0(\xi q_j)^2 + \frac{32}{25}M^4|\xi|^{3\alpha}\sum_{j=1}^n q_j^{3\alpha}\right)e^{\eta|\xi|^\alpha}\right\} \\ &\le e^{-(a_0-\eta)|\xi|^\alpha}|\xi|^\alpha\left\{\sum_{j=1}^n q_j^\alpha|v_0(\xi q_j)|\Big(1 + 2|\xi|^\alpha|v_0(\xi q_j)|\Big)\right. \\ &\quad \left. + |\xi|^\alpha M^2\sum_{j=1}^n q_j^{2\alpha}\left(\frac{4}{5} + \frac{32}{25}M^2|\xi|^{2\alpha}q_j^\alpha\right)\right\}.\end{aligned}$$

At this stage it remains to consider $|\xi| > D_n$. In this case, one gets

$$\begin{aligned}&\frac{|\xi|^s}{d_1^s}\left\{\frac{q_{(n)}^s}{d^{s/\alpha}}\mathbb{I}(c=0) + \frac{q_{(n)}^s}{d^{s/\alpha}}\mathbb{I}(q_{(n)} > d^{1/c\alpha}, 0 < c < 1) + q_{(n)}^{s(1-c)}\mathbb{I}(q_{(n)} \le d^{1/c\alpha}, 0 < c < 1)\right\} \\ &\quad \ge 1\end{aligned}$$

and, to complete the proof for (29), it is enough to take account of the obvious inequality $|\tilde{\phi}_n(\xi) - \exp(-a_0|\xi|^\alpha)| \le 2$.

Now, as far as (30) is concerned, take $s=\alpha$ and c in $(0,1)$. Then, (29) becomes

$$\begin{aligned}&|\tilde{\phi}_n(\xi)-\exp(-a_0|\xi|^\alpha)|\\&\quad\le e^{-(a_0-\eta)|\xi|^\alpha}|\xi|^\alpha\bar{\sigma}(\xi)\mathbb{I}\{|\xi|\le D_n\}\\&\qquad+2\frac{|\xi|^\alpha}{d_1^\alpha}\left\{\frac{q_{(n)}^\alpha}{d}\mathbb{I}(q_{(n)}>d^{1/c\alpha})+q_{(n)}^{\alpha(1-c)}\mathbb{I}(q_{(n)}\le d^{1/c\alpha})\right\}\mathbb{I}\{|\xi|>D_n\}.\end{aligned}\tag{34}$$

Now, for $q_{(n)}>d$ and $|\xi|\le D_n(\le d_1 d^{1/\alpha}q_{(n)}^{-1})$,

$$\bar{\sigma}(\xi)\le\bar{v}_0(d_1d^{1/\alpha})(1+2d_1^\alpha d^{1-\alpha}\bar{v}_0(d_1d^{1/\alpha}))+(4/5)M^2d_1^\alpha d+(32/25)M^4d_1^{3\alpha}d^{3-\alpha}=k^*$$

and (30) follows from (34) with $\bar{\sigma}(\xi)$ replaced by k^* on $\{q_{(n)}>d,|\xi|\le D_n\}$. □

Lemma 3.1 can be used to obtain bounds for the χ_α-distance between G_α and the probability distribution function F_n of S_n.

Proposition 3.2 *The χ_α-distance between F_n and G_α satisfies*

$$\begin{aligned}\chi_\alpha(F_n,G_\alpha)&\le k^*\mathbb{I}(q_{(n)}>d)+\sum_{j=1}^n q_j^\alpha\bar{v}_0\big(d_1q_jq_{(n)}^{c-1}\big)\big\{1+2M_1\bar{v}_0\big(d_1q_jq_{(n)}^{c-1}\big)\big\}\\&\quad+q_{(n)}^\alpha\left\{\frac{4}{5}M_1M^2+\frac{32}{25}M_3M^4q_{(n)}^\alpha\right\}+\frac{2}{d_1^\alpha}\left\{\frac{q_{(n)}^\alpha}{d}\mathbb{I}(q_{(n)}>d^{1/c\alpha})+q_{(n)}^{\alpha(1-c)}\right\}\end{aligned}$$

for any c in $(0,1)$, with $M_r:=\max_{x\ge0}e^{-(a_0-\eta)x^\alpha}x^{r\alpha}$ (r being any positive number).

Proof Consider (30) and observe that

$$\bar{\sigma}(\xi)\le\sum_{j=1}^n q_j^\alpha\bar{v}_0(d_1q_jq_{(n)}^{c-1})\Big(1+2M_1\bar{v}_0(d_1q_jq_{(n)}^{c-1})\Big)+\frac{4}{5}M^2M_1q_{(n)}^\alpha+\frac{32}{25}M^4M_3q_{(n)}^{2\alpha}$$

holds true on the set $\{q_{(n)}\le d,|\xi|\le D_n\}$ since $D_n\le d_1q_{(n)}^{c-1}$ on this set. □

It is easy to check that the upper bound stated in Proposition 3.2 is $o(1)$ for $n\to+\infty$.

Lemma 3.1 can also be exploited to determine analogous bounds for $\chi_{\alpha+\delta}$ and $\chi_{2\alpha}$, under the extra condition (12).

Proposition 3.3 *Suppose (12) is valid for some $\delta>0$ and take d in such a way that $|\xi|q_{(n)}\le d_1d^{1/\alpha}$ ($=q_{(n)}D_n$ if $c=0$) entails $\bar{v}_0(\xi q_j)\le\rho|\xi q_j|^\delta$ for some $\rho>0$. Then,*

$$\begin{aligned}\chi_{\alpha+\delta}(F_n,G_\alpha)&\le\rho\sum_{j=1}^n q_j^{\alpha+\delta}+2\rho^2M_{1+\frac{\delta}{\alpha}}\sum_{j=1}^n q_j^{\alpha+2\delta}\\&\quad+\frac{4}{5}M^2M_{1-\frac{\delta}{\alpha}}\sum_{j=1}^n q_j^{2\alpha}+\frac{32}{25}M^4M_{3-\frac{\delta}{\alpha}}\sum_{j=1}^n q_j^{3\alpha}+\frac{2q_{(n)}^{\alpha+\delta}}{d_1^{\alpha+\delta}d^{1+\delta/\alpha}}\end{aligned}$$

for any $\delta \le \alpha$, *and*

$$\chi_{2\alpha}(F_n, G_\alpha) \le \rho M_{\frac{\delta}{\alpha}-1} \sum_{j=1}^n q_j^{\alpha+\delta} + 2\rho^2 M_{\frac{2\delta}{\alpha}} \sum_{j=1}^n q_j^{\alpha+2\delta} + \frac{4M^2}{5} \sum_{j=1}^n q_j^{2\alpha} + \frac{32M^4 M_2}{25} \sum_{j=1}^n q_j^{3\alpha} + \frac{2q_{(n)}^{2\alpha}}{d_1^{2\alpha} d^2}$$

for any δ *in* $(\alpha, 2\alpha]$.

Proof From (29) with $c = 0$ and $s = \alpha + \delta$,

$$|\tilde{\phi}_n(\xi) - e^{-a_0|\xi|^\alpha}| \le e^{-(a_0-\eta)|\xi|^\alpha} |\xi|^\alpha \left\{ \rho \sum_{j=1}^n q_j^{\alpha+\delta} |\xi|^\delta (1 + 2\rho q_j^\delta |\xi|^{\alpha+\delta}) + |\xi|^\alpha M^2 \sum_{j=1}^n q_j^{2\alpha} \left(\frac{4}{5} + \frac{32}{25} M^2 q_j^\alpha |\xi|^{2\alpha} \right) \right\} \mathbb{I}(|\xi| \le d_1 d^{1/\alpha} q_{(n)}^{-1}) + \frac{2q_{(n)}^{\alpha+\delta}}{d_1^{\alpha+\delta} d^{1+\delta/\alpha}} |\xi|^{\alpha+\delta} \mathbb{I}(|\xi| > d_1 d^{1/\alpha} q_{(n)}^{-1}).$$

Then, if δ belongs to $(0, \alpha]$, one easily obtains the former of the inequalities to be proved. The latter follows similarly from (29) with $c = 0$ and $s = 2\alpha$. □

As mentioned at the beginning of Sect. 2.2, here we pass from weighted χ-metrics to Kolmogorov's metric via the classical Berry–Esseen inequality

$$K(F_n, G_\alpha) \le \frac{1}{\pi} \int_{-\tilde{d}/q_{(n)}}^{\tilde{d}/q_{(n)}} \left| \frac{\tilde{\phi}_n(\xi) - \hat{g}_\alpha(\xi)}{\xi} \right| d\xi + \frac{\mathbf{c}}{\tilde{d}} \|g_\alpha\| q_{(n)}$$

$\mathbf{c}$ being the constant which appears in Theorem 3.18 in [22].

Take (29), with $c = 0$ and $\tilde{d} = (3d/8M)^{1/\alpha}$, and substitute it in the right-hand side of the above Berry–Esseen inequality to obtain

Proposition 3.4 *One has*

$$K(F_n, G_\alpha) \le \frac{2}{\pi} \sum_{j=1}^n q_j^\alpha \int_0^{\tilde{d}/q_{(n)}} e^{-(a_0-\eta)\xi^\alpha} \xi^{\alpha-1} H(\xi, q_j) d\xi + \frac{8}{5\pi} M^2 N_{2\alpha} \sum_{j=1}^n q_j^{2\alpha} + \frac{64}{25\pi} M^4 N_{4\alpha} \sum_{j=1}^n q_j^{3\alpha} + \frac{\mathbf{c}}{\tilde{d}} \|g_\alpha\| q_{(n)} \tag{35}$$

with $H(\xi, q_j) := |v_0(\xi q_j)|(1 + 2|\xi|^\alpha |v_0(\xi q_j)|)$ *and* $N_l = \int_0^{+\infty} \exp\{-(a_0 - \eta)\xi^\alpha\} \xi^{l-1} d\xi$. *This upper bound is* $o(1)$ *as* $n \to +\infty$.

More informative bounds can be obtained under extra condition (12).

Proposition 3.5 *If* (12) *is valid for some* $\delta > 0$ *and* d *is fixed in such a way that* $|\xi| q_{(n)} \leq d_1 d^{1/\alpha}$ $(= q_{(n)} D_n$ *if* $c = 0)$ *entails* $v_0(\xi q_j) \leq \rho |\xi q_j|^\delta$ *for some* $\rho > 0$, *then*

$$K(F_n, G_\alpha) \leq \frac{2}{\pi}\Big[\rho N_{\alpha+\delta} \sum_{j=1}^n q_j^{\alpha+\delta} + 2\rho^2 N_{2(\alpha+\delta)} \sum_{j=1}^n q_j^{\alpha+2\delta} + \frac{4}{5} M^2 N_{2\alpha} \sum_{j=1}^n q_j^{2\alpha}$$
$$+ \frac{32}{25} M^4 N_{4\alpha} \sum_{j=1}^n q_j^{3\alpha}\Big] + \frac{\mathbf{c}}{\tilde{d}} \|g_\alpha\| q_{(n)} = o(1) \quad \textit{as } n \to +\infty.$$

Proof Under the present extra condition, inequality in the previous proposition combined with inequality $H(\xi, q_j) \leq \rho |\xi|^\delta q_j^\delta (1 + 2\rho q_j^\delta |\xi|^\delta)$, valid for every j and $|\xi| \leq \tilde{d}/q_{(n)}$, yields the desired bound. □

Now, we proceed to present bounds for $K(F_n, G_\alpha)$ under restrictions on the initial distribution function, rather than on ϕ_0. Notation is the same as in Sect. 2.3 with the proviso that F_0^* is replaced by (symmetric) F_0 and, consequently, symbols with $*$, like S^*, h^*, c_0^*, etc. must be changed over to symbols without $*$, i.e., S, h, c_0, etc., respectively.

Proposition 3.6 *Let* α *be in* $[1, 2)$ *and let the additional restriction that* $\int_0^{+\infty} |S(x)| dx < +\infty$ *if* $\alpha = 1$ *be valid. Then,*

$$K(F_n, G_\alpha) \leq \frac{2}{\pi} \sum_{j=1}^n \Big\{ B_1 q_j^2 + B_2 q_j^{4-\alpha} + (B_3 H_1(q_j) + B_4 H_2(q_j)) q_j^\alpha$$
$$+ B_5 q_j^{2\alpha} + B_6 q_j^{3\alpha} \Big\} + \frac{\mathbf{c}\|g_\alpha\|}{\tilde{d}} q_{(n)} = o(1) \quad \textit{as } n \to +\infty.$$

In particular, if h *is such that* $|h(x)| := x^\alpha |S(x)| \leq \rho'/x^\delta$ *for any* $x > 0$, δ *in* $(0, 2-\alpha)$ *and some constant* $\rho' > 0$, *then*

$$H_1(q) \leq \frac{\rho' q^\delta}{2 - \alpha - \delta}, \qquad H_2(q) \leq \frac{\rho' q^\delta}{\alpha + \delta - 1}$$

are valid for any q *in* $(0, 1]$.

Proof We start from the definitions of S and ϕ_0 to obtain, via (28),

$$1 - \phi_0(\xi) = a_0 |\xi|^\alpha + 2\xi \int_0^{+\infty} S(x) \sin(\xi x) dx$$

which, in view of (F_4), yields

$$|\xi|^\alpha |v_0(\xi q_j)| = \frac{1}{q_j^\alpha} |b_1(\xi q_j) + R_1(\xi q_j)|$$

where

$$b_1(y) := 2y \int_D^{+\infty} \sin(yx) S(x) dx \quad \text{and} \quad R_1(y) := 2y \int_0^D \sin(yx) S(x) dx.$$

For these quantities one can write

$$|R_1(\xi q_j)| \le 2\xi^2 q_j^2 \int_0^D x|S(x)|dx = 2k_1\xi^2 q_j^2$$

with $k_1 := \int_0^D x|S(x)|dx$, and

$$k_2 := \sup_{x>0} \frac{|b_1(x)|}{x^\alpha} \le \max\left\{ \|v_0\| + 2k_1, 2\int_D^{+\infty} |S(x)|dx \right\}.$$

Combination of these inequalities with the definition of H (see Proposition 3.4) gives us

$$\begin{aligned}
|\xi|^{\alpha-1}|H(\xi, q_j)| &= |\xi|^{\alpha-1}|v_0(\xi q_j)|(1 + 2|\xi|^\alpha |v_0(\xi q_j)|) \\
&\le \frac{1}{|\xi| q_j^\alpha} \left\{ |b_1(q_j\xi)| + |R_1(q_j\xi)| + \frac{2}{q_j^\alpha}\big(|b_1(q_j\xi)| + |R_1(q_j\xi)|\big)^2 \right\} \\
&\le \frac{1}{|\xi| q_j^\alpha} \Big\{ |b_1(q_j\xi)| + 2k_2|b_1(q_j\xi)||\xi|^\alpha + 2k_1|\xi q_j|^2 + 8k_1k_2 q_j^2|\xi|^{2+\alpha} \\
&\qquad + 8k_1^2 q_j^{4-\alpha}|\xi|^4 \Big\}.
\end{aligned}$$

Using this inequality, we obtain

$$\begin{aligned}
&\frac{2}{\pi}\sum_{j=1}^n q_j^\alpha \int_0^{\tilde{d}/q_{(n)}} e^{-(a_0-\eta)\xi^\alpha} \xi^{\alpha-1} H(\xi, q_j) d\xi \\
&\quad \le \frac{2}{\pi}\sum_{j=1}^n \bigg\{ \int_0^{+\infty} e^{-(a_0-\eta)\xi^\alpha} \Big(|b_1(q_j\xi)|\xi^{-1} + 2k_2|b_1(q_j\xi)|\xi^{\alpha-1} \Big) d\xi \\
&\qquad + 2k_1 N_2 q_j^2 + 8k_1k_2 q_j^2 N_{2+\alpha} + 8k_1^2 N_4 q_j^{4-\alpha} \bigg\}.
\end{aligned} \tag{36}$$

It remains to study integrals like $I_r(q) := \int_0^{+\infty} |b_1(\xi q)|\xi^{r-1} e^{-(a_0-\eta)\xi^\alpha} d\xi$ for $r \ge 0$. Following the argument used in [24] to prove Lemma 7, one can state the inequality

$$\begin{aligned}
I_r(q) &\le 2qN_{r+2}\int_{\frac{1}{q}}^{+\infty} |S(x)|dx + 2q^2 N_{r+1}\int_0^{\frac{1}{q}} x|S(x)|dx \\
&= 2N_{r+2}q^\alpha \int_1^{+\infty} |h(y/q)|y^{-\alpha}dy + 2N_{r+1}q^\alpha \int_0^1 |h(y/q)|y^{1-\alpha}dy \\
&= 2N_{r+2}q^\alpha H_2(q) + 2N_{r+1}q^\alpha H_1(q)
\end{aligned} \tag{37}$$

with $h(x) = x^\alpha S(x)$. To complete the proof of the main part of the proposition it is enough to use (37) to obtain a bound for the right-hand side of (36) and, then, to replace this bound for the first sum in the right-hand side of (35). As to the latter claim, recall that $H_1(q) = \int_0^1 y^{1-\alpha}|h(y/q)|dy$, $H_2(q) = \int_1^{+\infty} y^{-\alpha}|h(y/q)|dy$ and use the additional condition. □

Proposition 3.7 *Let α be in $(0,2)$ and let the additional hypothesis that S is monotonic on $[D,+\infty)$ be valid for some $D \geq 0$. Then,*

$$K(F_n, G_\alpha) \leq \frac{2}{\pi}\sum_{j=1}^{n}\Big\{\bar{B}_1 q_j^2 + \bar{B}_2 q_j^{4-\alpha} + \bar{B}_3(H_1(q_j)+H_3(q_j))q_j^\alpha + B_5 q_j^{2\alpha} + B_6 q_j^{3\alpha}\Big\} + \frac{\mathbf{c}\|g_\alpha\|}{\tilde{d}} q_{(n)} = o(1) \quad \text{as } n \to +\infty.$$

Moreover, if h is such that $|h(x)| \leq \rho'/x^\delta$ for any $x > 0$, δ in $(0, 2-\alpha)$ and some constant $\rho' > 0$, one gets

$$H_1(q) \leq \frac{\rho' q^\delta}{2-\alpha-\delta}, \qquad H_3(q) \leq \frac{\rho' q^\delta}{\alpha+\delta}$$

for every q in $(0,1]$.

Proof One starts from Proposition 3.4 once again, noticing that equality

$$|t|^\alpha v_0(t) = b_2(t) + R_2(t)$$

holds with

$$b_2(t) := -2\int_D^{+\infty}(1-\cos(tx))dS(x) \quad \text{and} \quad R_2(t) := R_1(t) + 2S(D)(\cos(tD)-1).$$

Observe that

$$|R_2(\xi q_j)| \leq 2\bar{k}_1|\xi q_j|^2$$

with $\bar{k}_1 = k_1 + D^2|S(D)|/2$. Moreover,

$$\bar{k}_2 = \sup_{x>0}\frac{|b_2(x)|}{x^\alpha} \leq k_2 + 2D|S(D)|\max\Big(\frac{D^2}{2}, 2\Big).$$

Then,

$$\begin{aligned}|\xi|^{\alpha-1}|H(\xi,q_j)| &\leq \frac{1}{|\xi|q_j^\alpha}\Big\{|b_2(\xi q_j)| + |R_2(\xi q_j)| + 2\frac{(|b_2(\xi q_j)|+|R_2(\xi q_j)|)^2}{q_j^\alpha}\Big\} \\ &\leq \frac{1}{|\xi|q_j^\alpha}\{|b_2(\xi q_j)| + 2|\xi|^\alpha\bar{k}_2|b_2(\xi q_j)| + 2\bar{k}_1|\xi q_j|^2 + 8\bar{k}_1^2 q_j^{4-\alpha}|\xi|^4 \\ &\quad + 8\bar{k}_1\bar{k}_2 q_j^2|\xi|^{\alpha+2}\}.\end{aligned}$$

Hence,

$$\begin{aligned}K(F_n,G_\alpha) \leq \frac{2}{\pi}\sum_{j=1}^{n}\Big\{&\int_0^{\tilde{d}/q_{(n)}}\frac{e^{-(a_0-\eta)\xi^\alpha}}{\xi}[1+2\bar{k}_2\xi^\alpha]|b_2(\xi q_j)|d\xi \\ &+ (2\bar{k}_1 N_2 + 8\bar{k}_1\bar{k}_2 N_{\alpha+2})q_j^2 + 8\bar{k}_1^2 N_4 q_j^{4-\alpha} + \frac{4}{5}M^2 N_{2\alpha}q_j^{2\alpha} + \frac{32}{25}M^2 N_{4\alpha}q_j^{3\alpha}\Big\} \\ &+ \frac{\mathbf{c}\|g_\alpha\|}{\tilde{d}}q_{(n)}.\end{aligned}$$

Applying the Fubini theorem and the formula for integration by parts, we can write

$$\begin{aligned}
\mathcal{M}_r(q) &:= \int_0^{\tilde{d}/q_{(n)}} n_r(\xi)\frac{|b_2(\xi q)|}{\xi}d\xi \quad (\text{with } n_r(\xi) := e^{-(a_0-\eta)\xi^\alpha}\xi^r) \\
&\le 2\left|S(D)\int_0^{\tilde{d}/q_{(n)}}(1-\cos(\xi q D))\frac{n_r(\xi)}{\xi}d\xi\right| \\
&\quad + 2q\left|\int_D^{+\infty}S(x)dx\int_0^{\tilde{d}/q_{(n)}}n_r(\xi)\sin(\xi q x)d\xi\right| \\
&\le |S(D)|q^2D^2N_{r+2} + \mathcal{M}_r^{(1)}(q)
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{M}_r^{(1)}(q) &:= 2q\left|\int_D^{+\infty}S(x)dx\int_0^{\tilde{d}/q_{(n)}}n_r(\xi)\sin(\xi q x)d\xi\right| \\
&\le 2q\int_D^{+\infty}\frac{|S(x)|}{x}dx\int_0^{+\infty}(1-\cos(\xi q x))\left|\frac{d}{d\xi}n_r(\xi)\right|d\xi \\
&\quad \text{(from integration by parts)} \\
&\le 2\int_D^{+\infty}\frac{|S(x)|}{x}\int_0^{+\infty}\left(1\wedge\frac{(\xi q x)^2}{2}\right)\left|\frac{d}{d\xi}n_r(\xi)\right|d\xi \\
&\le 2z_r\left\{q^2\int_0^{1/q}x|S(x)|dx + \int_{1/q}^{+\infty}\frac{|S(x)|}{x}dx\right\} \\
&\quad \left(\text{with } z_r := \max\left\{\int_0^{+\infty}\left|\frac{d}{d\xi}n_r(\xi)\right|d\xi, \frac{1}{2}\int_0^{+\infty}\xi^2\left|\frac{d}{d\xi}n_r(\xi)\right|d\xi\right\}\right) \\
&= 2z_r\{q^\alpha H_1(q) + q^\alpha H_3(q)\}.
\end{aligned}$$

Then,

$$\mathcal{M}_r(q) \le q^2 \mid S(D) \mid D^2N_{r+2} + 2q^\alpha z_r\{H_1(q)+H_3(q)\}$$

and

$$\begin{aligned}
K(F_n, G_\alpha) \le \frac{2}{\pi}\sum_{j=1}^n\Bigg\{&\mathcal{M}_0(q_j) + 2\bar{k}_2\mathcal{M}_\alpha(q_j) + (2\bar{k}_1N_2 + 8\bar{k}_1\bar{k}_2N_{\alpha+2})q_j^2 \\
&+ 8\bar{k}_1^2N_4q_j^{4-\alpha} + \frac{4}{5}M^2N_{2\alpha}q_j^{2\alpha} + \frac{32}{25}M^2N_{4\alpha}q_j^{3\alpha}\Bigg\} + \frac{\mathbf{c}\|g_\alpha\|}{\tilde{d}}q_{(n)}.
\end{aligned}$$

To complete the proof suffices it to replace the quantities $\mathcal{M}$ with their upper bounds and, next, to recall the definition of the constants $\bar{B}$. □

Appendix

In this part of the paper we present the proofs of the theorems stated in Sect. 2. For the sake of expository clarity, let us recall the common inspiring principles for all of these proofs.

First of all, we refer to representation (5) which, combined with (15), gives

$$|\phi(\xi,t)-\hat{g}_\alpha(\xi)| \le E_t(|\tilde{\phi}_{\nu_t}(\xi;Re(\phi_0))-\hat{g}_\alpha(\xi)|)+|Im(\phi_0(\xi))|e^{-t} \quad (\xi\in\mathbb{R}) \tag{38}$$

where $\tilde{\phi}_{\nu_t}(\cdot\,;Re(\phi_0))$ is equal to $\tilde{\phi}_n(\cdot)$ when $n=\nu_t$, $q_j=|\beta_{j,t}|$ $(j=1,\dots,\nu_t)$ and, in the definition of $\tilde{\phi}_n$, ϕ_0 is replaced by $Re\phi_0$. Analogously,

$$|F(x,t)-G_\alpha(x)| \le E_t(|F_{\nu_t}(x;F_0^*)-G_\alpha(x)|)+|F_0(x)-F_0^*(x)|e^{-t} \quad (x\in\mathbb{R}) \tag{39}$$

where $F_{\nu_t}(\cdot\,;F_0^*)$ is obtained from $F_n(\cdot)$ by replacing n, q_j and F_0 with ν_t, $|\beta_{j,t}|$ and F_0^*, respectively.

Proof of Theorem 2.2 Apply (38) to write

$$\chi_\alpha(F(\cdot,t),G_\alpha) \le E_t(\chi_\alpha(F_{\nu_t}(\cdot\,;F_0^*),G_\alpha))+e^{-t}\sup_{\xi\in\mathbb{R}}|Im(v_0(\xi))|$$

and, next, replace $\chi_\alpha(F_{\nu_t}(\cdot\,;F_0^*),G_\alpha)$ with its upper bound stated in Proposition 3.2. □

Proof of Theorem 2.3 Argue as in the previous proof by using the upper bounds obtained in Proposition 3.3, instead of the upper bound of Proposition 3.2. Moreover, to evaluate expectations, make use of the obvious inequality $E_t(\beta^m_{(\nu_t)}) \le E_t[\sum_{j=1}^{\nu_t}|\beta_{j,t}|^m]$ and, then, of (19) and (20). □

Proof of Theorem 2.4 In view of (39), write

$$K(F(\cdot,t),G_\alpha) \le E_t(K(F_{\nu_t}(\cdot\,;F_0^*),G_\alpha))+\frac{e^{-t}}{2}\sup_{x\in\mathbb{R}}|F_0(x)+F_0(-x-0)-1|$$

and replace $K(F_{\nu_t}(\cdot\,;F_0^*),G_\alpha)$ with its upper bound determined in Proposition 3.4. Finally use (19) to evaluate expectation. □

The remaining theorems from 2.5 to 2.9 can be proved following the same line of reasoning, according to the scheme: Resort to Proposition 3.5 and to (19) for Theorem 2.5. Apply Proposition 3.6 and (19)–(20) to prove Theorems 2.6 and 2.7. Finally, use Proposition 3.7 and (19)–(20) to prove Theorems 2.8 and 2.9.

It remains to prove Theorem 2.1. Its former part is a straightforward consequence of Theorem 2.4. As to the latter, we use the same argument as in the proof of Theorem 1 in [19], based on [16]. Accordingly, for every $t>0$ we define

$$W_t := (\Lambda_{\nu_t},\lambda_{1,t},\dots\lambda_{\nu_t,t},\delta_0,\dots,\gamma_t,\theta_t,\nu_t,U_t(1/2),U_t(1/3),\dots)$$

where: $\lambda_{j,t}$ stands for a conditional distribution of $|\beta_{j,t}|X^*_{j,t}$, given $(\gamma_t,\theta_t,\nu_t)$; Λ_{ν_t} is the ν_t-fold convolution of $\lambda_{1,t},\dots\lambda_{\nu_t,t}$; δ_x indicates unit mass at x; $U_t(\xi) := \max_{1\le j\le\nu_t}\lambda_{j,t}([-\xi,\xi]^c)$. Moreover, the $X^*_{j,t}$ are conditionally i.i.d. with common distribution F_0^*. To grasp the importance of W_t, notice that its components represent the essential ingredients of central limit problems. As to this fundamental theorem, we refer to Sect. 16.8 of [17]. The range of W_t can be seen as a subset of $S := \mathbb{P}(\bar{\mathbb{R}})^\infty\times\bar{\mathbb{G}}\times[0,2\pi)^\infty\times\bar{\mathbb{R}}^\infty$, where: $\bar{\mathbb{R}} := [-\infty,+\infty]$; $\mathbb{P}(M)$ indicates the set of all probability measures on the Borel class $\mathcal{B}(M)$ on some metric space M; $\bar{\mathbb{G}}$ is a distinguished metrizable compactification of $\mathbb{G}$.

These spaces are endowed with topologies specified in Sect. 3.2 of [19], which make S a separable compact metric space. Now recall that, under the assumption of the latter part of Theorem 2.1, $(V_{t_n}^* := \sum_{j=1}^{\nu_{t_n}} |\beta_{j,t_n}| X_{j,t_n}^*)_{n\geq 1}$ must converge in distribution. Next, from Lemma 3 in [19], with slight changes, the sequence of the laws of the vectors $(W_{t_n})_{n\geq 1}$ contains a subsequence $(W_{t_{n'}})_{n'}$ which is weakly convergent to a probability measure Q supported by $\mathbb{P}(\mathbb{R}) \times \{\delta_0\}^\infty \times \bar{\mathbb{G}} \times [0, 2\pi)^\infty \times \{+\infty\} \times \{0\}^\infty$. At this stage, an application of the Skorokhod representation theorem (see, e.g., [1, 14]), combined both with the properties of the support of Q and with (F_1), entails the existence of random vectors $\hat{W}_{t_{n'}} := (\hat{\Lambda}_{\hat{\nu}_{t_{n'}}}, \hat{\lambda}_{1,t_{n'}}, \dots)$ defined on a suitable space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P})$, in such a way that $W_{t_{n'}}$ and $\hat{W}_{t_{n'}}$ have the same law (for every n'). Moreover,

$$
\begin{aligned}
&\hat{\Lambda}_{\hat{\nu}_{t_{n'}}} \Rightarrow \hat{\Lambda}, \qquad \hat{\lambda}_{j,t_{n'}} \Rightarrow \delta_0 \quad (j = 1, 2, \dots) \\
&\hat{\nu}_{t_{n'}} \to +\infty, \qquad \hat{U}_{t_{n'}}(1/k) \to 0 \quad (k = 1, 2, \dots), \\
&\hat{\beta}_{(n')} := \max\{|\hat{\beta}_{1,t_{n'}}|, \dots |\hat{\beta}_{\hat{\nu}_{t_{n'}},t_{n'}}|\} \to 0
\end{aligned}
\tag{40}
$$

where the convergence must be understood as pointwise convergence on $\hat{\Omega}$ and $\Rightarrow$ designates weak convergence of probability measures. From (40) and Theorem 16.24 of [17], there is a random Lévy measure μ, symmetric about zero, such that

$$
\lim_{n'\to+\infty} \sum_{j=1}^{\hat{\nu}_{t_{n'}}} \hat{\lambda}_{j,t_{n'}}[x, +\infty) = \lim_{n'\to+\infty} \sum_{j=1}^{\hat{\nu}_{t_{n'}}} \left\{1 - F_0^*\left(\frac{x}{|\hat{\beta}_{j,t_{n'}}|}\right)\right\} = \mu[x, +\infty) \tag{41}
$$

holds pointwise on $\hat{\Omega}$ for every $x > 0$. To complete the proof, we assume that $\lim_{x\to+\infty} x^\alpha\{1 - F_0^*(x)\} = +\infty$ and show that this assumption contradicts (41). Indeed, the assumption implies that for any $k > 0$ there is $\epsilon > 0$ such that $x^\alpha\{1 - F_0^*(x)\} \geq k$ for every $x > 1/\epsilon$ and, therefore,

$$
\begin{aligned}
\nu_{n,x} &:= \sum_{j=1}^{\hat{\nu}_{t_{n'}}} \left\{1 - F_0^*\left(\frac{x}{|\hat{\beta}_{j,t_{n'}}|}\right)\right\} \\
&\geq \frac{k}{x^\alpha} \mathbb{I}\{\hat{\beta}_{(n')} < x\epsilon\} \sum_{j=1}^{\hat{\nu}_{t_{n'}}} |\hat{\beta}_{j,t_{n'}}|^\alpha \\
&= \frac{k}{x^\alpha} \mathbb{I}\{\hat{\beta}_{(n')} < x\epsilon\}.
\end{aligned}
$$

Since (40) yields $\hat{\beta}_{(n')} \to 0$, then $\limsup_{n\to+\infty} \nu_{n,x} \geq kx^{-\alpha}$, which contradicts (41) in view of the arbitrariness of k.

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